# MTH 508/608: Introduction to Differentiable Manifolds and Lie Groups
## Assignment 1

# 1 Practice problems

## 1.1 Differential multivariable calculus and topological manifolds

1. Review the proofs of all the assertions stated in Subsection 1.1 of the Lesson Plan.

2. Establish the assertions in 1.2 (iii)(e) and 1.2 (iv) of the Lesson Plan.

3. Show that conditions (a) and (b) of Definition 1.2 (v) of the Lesson Plan are mutually exclusive.

4. Show that the space obtained from the unit sphere $S^2$ in $\mathbb{R}^3$ by identifying the antipodal points and the space of all planes through the origin in $\mathbb{R}^3$ are both homeomorphic to $\mathbb{R}P^2$.

5. Show that the space $X$ of all orthonormal pairs of vectors in $\mathbb{R}^3$ is a manifold that is homeomorphic to the unit tangent bundle of the sphere $S^2$ (comprising all unit vectors tangent to $S^2$) .

6. Show that the space of all vectors in $\mathbb{R}^3$ normal to a curve (one-manifold) $C$ in $\mathbb{R}^3$ is a 3-manifold.

7. Show that $S^1 \times S^1$ and $S^2 \times S^1$ are 3-manifolds.

## 1.2 Smooth manifolds and mappings

1. Give a $C^\infty$ structure on $S^n$ consisting of two coordinate neighborhoods using the stereographic projections from the north and south poles.

2. Show that the quotient map $p : S^n \to \mathbb{R}P^n$ obtained by identifying the antipodal pairs of points in $S^n$ is $C^\infty$ of constant rank $n$.

3. Let $M$ and $N$ be smooth manifolds, $U \subset M$ is open, and $f : U \to N$ a $C^\infty$ mapping. Show that there exists a neighborhood $V(\subset U)$ of any $p \in U$ such that $f$ can be extended to a $C^\infty$ mapping $f^* : M \to N$ with $f(x) = f^*(x)$ for all $x \in V$.

4. Let $\mathcal{M}_{mn}(\mathbb{R})$ be the space of all real $m \times n$ matrices over $\mathbb{R}$, and $\mathcal{M}^k_{mn}(\mathbb{R}) \subset \mathcal{M}_{mn}(\mathbb{R})$ be the subset of matrices whose rank $\geq k$. Show that $\mathcal{M}^k_{mn}(\mathbb{R})$ is an open subset of $\mathcal{M}_{mn}(\mathbb{R})$ and hence a smooth manifold.

5. Let $M, N$ be smooth manifolds, and let $f : N \to M$ be a submersion.
   (a) Show that $f$ is an open map.
   (b) If $\dim(M) = \dim(N)$, then show that $f$ is a local (not global) diffeomorphism onto its image.
   (c) Given an example of a submersion $f$ that is not a (global) diffeomorphism onto its image.

6. Let $G(k, n)$ be the Grassman manifold described in Example 1.2(v)(g) of the Lesson Plan.
   (a) Show that the quotient map $\pi : F(k, n) \to G(k, n)$ is open and $G(k, n)$ is Hausdorff
   (b) Fill in the other details in the argument to show that $G(k, n)$ is a smooth manifold.

7. Show that the reflection about the $x$-axis of the figure eight curve in Example 1.2.3 (xvii)(c) of the Lesson Plan is not a diffeomorphism.

8. For $i = 1, 2$, two injective immersions $f_i : N_i \to M$ of smooth manifolds are *equivalent* if there exists a diffeomorphism $g : N_1 \to N_2$ such that $f_1 = f_2 \circ g$.
   (a) Show that this equivalence is an equivalence relation.
   (b) Find two inequivalent injective immersions $\mathbb{R} \to \mathbb{R}^2$.

9. If $f : N \to M$ be be a $C^\infty$ mapping of smooth manifolds and $A \subset N$ is an immersed submanifold of $N$, then $f|_A$ is a $C^\infty$ mapping into $M$.

10. Let $f : N \to M$ be an injective immersion. Then show that $f$ is proper (i.e., inverse of a compact set is compact) if and only if $f$ is an imbedding and $f(N)$ is a closed regular submanifold of $M$.

11. If $N$ is a submanifold of a smooth manifold $M$ and $V \subset M$ is open, then show that $N \cap U$ is a countable union of connected open subsets of $N$.

12. Give an example of a $C^\infty$ function on a submanifold $N$ of a smooth manifold $M$ that is not the restriction of a $C^\infty$ function on $M$.

## 1.3 Lie groups and their actions

1. Let $G$ be a Lie group and let $e \in G$ be its identity.

   (a) Show that given any neighborhood $U \ni e$ there exists a neighborhood $V \ni e$ such that $VV^{-1} \subset U$.

   (b) Show that given any neighborhood $U \ni e$ there exists a neighborhood $W \ni e$ such that $W^2 = WW \subset U$.

2. Let $G$ be a Lie group. Show that if $A \subset G$ and $U \subset G$ is open, then $AU$ is open in $G$.

3. Show that if $H$ is an algebraic subgroup of a Lie group $G$, then $\overline{H}$ is also an algebraic subgroup of $G$.

4. Establish the assertions in Example 1.3.1 (viii)(c) of the Lesson Plan. Furthermore, show that is $\alpha$ is rational, then $f(L_\alpha)$ is a regular submanifold of $T^2$.

5. Establish the assertions in Example 1.3.2 (v)(c) of the Lesson Plan.

6. Show that the Lie group $\mathrm{O}(n, \mathbb{R})$ has a natural action on $S^{n-1}$ that is transitive. Find the stabilizer (subgroup) of $(1, 0, \ldots, 0)$ under this action.

7. Show that the Lie group $\mathrm{GL}(n, \mathbb{R})$ has a natural action on $\mathbb{R}P^{n-1}$ that is transitive. Find the stabilizer (subgroup) of $(1, 0, \ldots, 0)$ under this action.

8. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R} \text{ and } a > 0 \right\}.$$

(a) Show that $G$ is a Lie group.

(b) Show that the map $\theta : G \times \mathbb{R} \to \mathbb{R}$ defined by

$$\theta\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, x\right) = ax + b$$

defines an action.

(c) Does $\theta$ define a transitive action?

9. Consider the map $\theta : \mathbb{R}^* \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ defined $\theta(t, x) = tx$.

   (a) Show that $\theta$ defines an action of $\mathbb{R}^*$ on $\mathbb{R}^{n+1}$.

   (b) Under this action, show that the orbit space $\mathbb{R}^{n+1}/\mathbb{R}^* \approx \mathbb{R}P^n$.

# 2 Problems for submission

- **Homework 1 (Due 17/9/24)**: Solve problems 1.1 - 5, 6 and 1.2 - 2, 4, 6 & 10 from the practice problems above.
- **Homework 2 (Due 27/9/24)**: Solve problems 1.3 - 1, 4, 7 & 9 from the practice problems above.